Recitation 1

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1 Return of the Pigs

Recall Pass the Pigs from last week. Note the new column denoting point values for each roll.

Roll Type	Number of Rolls	Proportion	Point Value
Blank	1,344	.341	0
Dot	1,294	.329	1
Razorback	767	.195	5
Trotter	365	.092	5
Snouter	137	.035	5
Leaning Jowler	32	.008	10

Table 1: Empirical Likelihood of Pig Roll Types

What is the expected value of a roll in this game? Variance?
The expected value is:

$$.341(0) + .329(1) + .195(5) + .092(5) + .035(5) + .008(10) \approx 2$$

We'll use the quick formula for the variance, $\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$, for which we need to calculate $\mathbb{E}[X^2]$:

$$.341(0^2) + .329(1^2) + .195(5^2) + .092(5^2) + .035(5^2) + .008(10^2) \approx 9$$

So $\mathbb{V}[X] \approx 9 - 4 \approx 5$.

2 Backgammon

In backgammon, your turn starts with a roll of two dice. You are able to move your chips as many spaces as the sum of the two faces, with one backgammon-specific twist. Doubles count double. So, for example, rolling 4-4 means you can move $4 + 4 + 4 + 4 = 16$ total spaces, but a roll of 1-6 means that you can move $6 + 1 = 7$ spaces.

1. What is the expected number of spaces you can move on a single turn in Backgammon?
2. A backgammon board is 24 spaces long. How many turns do you expect it to take to make a full run for two chips?

1. We know the average of a standard dice roll is 7, which is from $2 + 3 + 3 + \ldots + 11 + 11 + 12$ (adding all possible outcomes) divided by 36 (the number of possible outcomes), so the sum must be $7 \cdot 36 = 252$. To this total, we must add the total pips that are bonused as a result of rolling doubles: $2+4+6+8+10+12 = 42$ (when we roll 1-1, we get $1+1+1+1$, but the first two of these were taken care of in arriving to 252; the latter two were as yet uncounted, and are now covered by the 2 in $2+\ldots+12$).

   Thus the numerator is now $252 + 42 = 294$, so the average is $\frac{294}{36} \approx 8.17$ – slightly higher than 7, as we might have anticipated.
2. Two full chips require 48 total spaces, or $approx \frac{48}{\frac{294}{36}} \approx 6$.

3 Linear Combinations

Consider the discrete RVs X and Y with PMFs:

x	-1	0	1
$\mathbb{P}[X = x]$	.05	.4	.55

Table 2: PMF of X

y	2	3	5	7	11
$\mathbb{P}[Y = y]$	.1	.1	.1	.1	.6

Table 3: PMF of X

1. Describe the distributions of X and Y at a glance.
2. Draw the CDFs of X and Y.
3. What is $\mathbb{E}[X]$?
4. What is $\mathbb{E}[X^2]$?
5. What is $\mathbb{V}[Y]$?
6. What is $\mathbb{E}[X^2 - \frac{1}{2}Y]$?
7. What is $\mathbb{E}[Z \mid Z > 5]$, where $Z = X + |Y - 5|$

1. X is highly concentrated on 1; Y is highly concentrated on 11, but uniform on other values.

2. Step functions

3. $.05(-1) + .4(0) + .55(1) = .5$

4. $.05(1) + .55(1) = .6$

5. $\mathbb{E}[Y] = .1(2+3+5+7)+.6(11) = 8.3$; $\mathbb{E}[Y^2] = .1(4+9+25+49)+.6(121) = 81.3$, so $\mathbb{V}[Y] = 81.3 - 8.3^2 = 12.41$

6. $\mathbb{E}[X^2 - \frac{1}{2}Y] = \mathbb{E}[X^2] - \frac{1}{2}\mathbb{E}[Y] = .6 - \frac{1}{2}8.3 = -3.55$

7. Technically, we need to know that X and Y are independent, which I forgot to mention. $X+|Y-5|$ can only more than 5 if $Y = 11$ and ($X = 0$ or $X = 1$). So if Z is more than 5, its expected value is: $\frac{.4}{.4+.55}(6)+\frac{.55}{.4+.55}(7) \approx 6.6$.

4 Continuous Distribution

Suppose that the random variable X has pdf

$$p(x) = \frac{1}{2}(x-1)\mathbb{1}[1 < x < 3]$$

(the $\mathbb{1}[1 < x < 3]$ part is just a convenient way of expressing that the domain of this RV is $[1, 3]$ in-line instead of needing to define it piecewise)

1. Find a monotone function $u(x)$ such that the random variable $Y = u(x)$ is distributed $U[0, 1]$.

2. Think of how you'd write a computer program to generate draws from the random variable X. *Hint: you can start from $U[0, 1]$ draws and then convert them into draws of X.*

$u = F^{-1}(x)$, where F^{-1} is the inverse of the CDF of X.