Recitation 5

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Welfare Maximization

Suppose (throwing an entire course in graduate-level economics into a black box) that a society's preferences over schools, S, and parks, P, are given by the following *welfare function*:

$$W(S,P) = \frac{3}{4}\ln S + \frac{1}{4}\ln P$$

Suppose further that the government has \$100 million to allocate between these two uses of public funds.

Each school costs \$25 million, and each park costs \$5 million.

What is the welfare-maximizing allocation of schools and parks?

The formal problem that we're solving is

$$\underset{S,P}{\text{Max}}\left\{W(S,P)\right\} \qquad \text{s.t. } 25S + 5P \leq 100; \qquad S, P \geq 0$$

We'll focus on the final part of solving this problem. First notice that, since welfare is increasing in both S and P, the government will spend all \$100 million (in more common terms – society is improved by any additional school or park, so there is no incentive not to spend all of the money).

In math, this means that

$$25S + 5P = 100$$

(Be sure you understand what this equation means!)

With this, we can eliminate one of the two choice variables, say P, by replacing P everywhere with $P = 20 - 5S$. Thus the problem simplifies to:

$$\underset{S}{\text{Max}}\left\{\frac{3}{4}\ln S + \frac{1}{4}\ln\left(20 - 5S\right)\right\}$$

We're trying to maximize the function $W(S) \equiv \frac{3}{4}\ln S + \frac{1}{4}\ln\left(20 - 5S\right)$ with respect to S. What on earth does this function look like?

Can plot with R or Wolfram Alpha, with output in Figure 1:

Our goal is to identify analytically the exact place where this function is maximized – visually, it's in the neighborhood of $S = 3$.

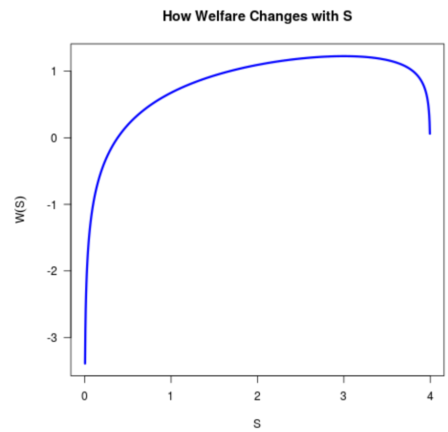

Figure 1: Appearance of Welfare Function at Budget Line

1. What is the analytic/calculus condition that describes the value of S that achieves the maximum of this function?
2. What is the derivative of W with respect to S?
3. What value of S eliminates the derivative (i.e., for which S is $W'(S) = 0$)?
4. *(Complete in your free time)* What is the second derivative of W with respect to S, and what can we learn from it about the critical point discovered in part 3)?

Some Matrix Algebra

Express the following system of equations as a single matrix equation of the form $AX = B$:

$$\begin{aligned} 2x + y + 2z &= 10 \\ x + y + z &= 6 \\ x + 3y + 2z &= 13 \end{aligned}$$

The solution to the matrix equation $AX = B$ is given by $X = A^{-1}B$.

1. Find A^{-1}.
2. Find $A^{-1}B$.