Intermediate Micro HW 2

June 13, 2016

1 Leontief & Substitution

An individual has Leontief preferences over goods x_1 and x_2. He starts with income y and the two goods have respective prices p_1 and p_2.

The price of good x_2 increases to p_2'. Decompose this individual's change in demand into income and substitution effects.

Leontief means utility can be represented by $u(x_1, x_2) = \min\{\alpha x_1, x_2\}$.

We know that demand for perfect complements comes in lockstep – individuals are only willing to purchase bundles along the corner of the indifference curves, which is given by the line $\alpha x_1 = x_2$. They then buy as many "pairs" of the goods as possible:

$$p_1 x_1 + p_2 x_2 = p_1 x_1 + \alpha p_2 x_1 = (p_1 + \alpha p_2) x_1 = y$$

So $x_1 = \frac{y}{p_1 + \alpha p_2}$, $x_2 = \alpha x_1 = \frac{\alpha y}{p_1 + \alpha p_2}$.

When p_2 increases to p_2', demand for good 2 changes from $\frac{\alpha y}{p_1 + \alpha p_2}$ to $\frac{\alpha y}{p_1 + \alpha p_2'}$. To decompose this into substitution and income effects, we first leave purchasing power constant and look at the change in demand in the absence of income effects, then we change income to reflect the change in demand in the absence of relative price changes.

Substitution Effect

If we simply rotate the new budget line through the same corner of the ICs (which is in effect what we do when we rotate the demand curve through the originally demanded bundle), demand will not change. This is the hallmark of perfect complements. As long as we can still purchase the same number of "pairs", we will.

Thus the substitution effect is 0. See Figure 1.

Income Effect

Since the substitution effect is 0, the entire change in demand is due to the income effect.

We want to buy as many "pairs" as we can; the increase in the price of good 2 has reduced our effective purchasing power, thereby reducing the number of "pairs" we can afford. It is for this reason and this reason alone that demand shifts. Again, see Figure 1
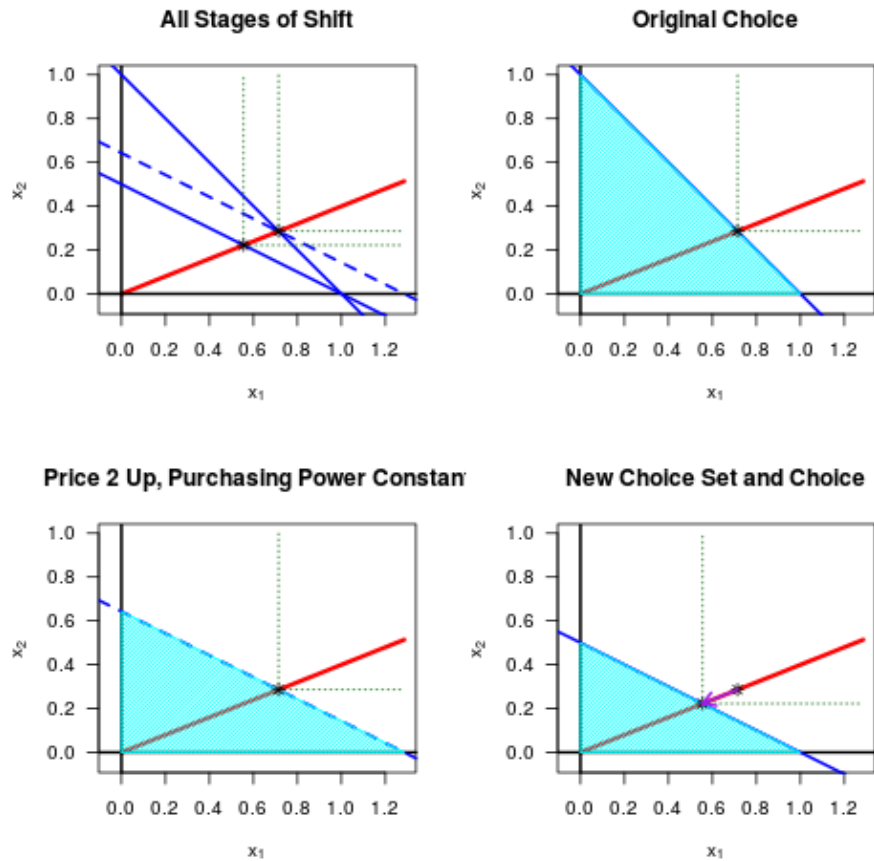

Figure 1: Progression of Demand Shift under Leontief

2 CES Demand

Another commonly used class of utility functions is CES utility, which stands for **constant elasticity of substitution**. In general they take the form:

$$u(x_1, x_2) = (\theta x_1^\rho + (1 - \theta)x_2^\rho)^{\frac{1}{\rho}}$$

If we take $\theta = \frac{1}{2}$, these are equivalent to

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$$

1. Derive the demand for CES utility with $\theta = \frac{1}{2}$, i.e., find $x_1(p_1, p_2, y)$ and $x_2(p_1, p_2, y)$.

   For simplicity, I'll normalize $p_2 = 1$. Remember that demand is only defined relatively – if we double all prices and income, the set of things we can afford is the same, and so demand will be the same. If we allow p_2 to vary freely, we'd replace all instances of p_1 in what follows with $\frac{p_1}{p_2}$, and all instances of y with $\frac{y}{p_2}$. Essentially, we're defining all "monetary" units in terms of good two.

   With that in mind, we proceed by setting the MRS equal to the price ratio (which is now just p_1):

   $$\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = \left(\frac{x_1}{x_2}\right)^{\rho-1} = p_1$$

   This gives us an expression for the ratio of goods that a CES individual will choose as a function of prices. We need to include information about income to pin down exact quantities. To do this, we solve this expression for x_1 and plug it into the budget constraint, which will yield the demand for x_2 as a function of p_1 and y:

   $$p_1 x_1 + x_2 = \left(1 + p_1^{\frac{\rho}{\rho-1}}\right) x_2 = y$$

   From which we determine $x_2 = \frac{y}{1+p_1^{\frac{\rho}{\rho-1}}}, x_1 = \frac{y p_1^{\frac{1}{\rho-1}}}{1+p_1^{\frac{\rho}{\rho-1}}}$.

2. Decompose a change in the price of good 1 from p_1 to p_1' into substitution and income effects.

   Now is when things get messy.

   The final demand for good 1 is given by $x_1' = \frac{y\left(p_1'\right)^{\frac{1}{\rho-1}}}{1+\left(p_1'\right)^{\frac{\rho}{\rho-1}}}$.

   The intermediate demand for good 1 we'll call x_1^s. This is the amount of good one we choose when p_1' is implemented, but we can still afford x_1 and x_2. That is,

   $$p_1 x_1 + x_2 = p_1' x_1 + x_2 = y \frac{\frac{p_1'}{p_1} p_1^{\frac{\rho}{\rho-1}} + 1}{p_1^{\frac{\rho}{\rho-1}} + 1}$$

   This essentially gives us a new income, y', which we can plug into the demand for x_1. For simplicity, we'll call the constant multiplying y in y' λ, so $y' = \lambda y$:

$$x_1^s = \frac{\lambda y \left(p_1^{'}\right)^{\frac{1}{\rho-1}}}{\left(p_1^{'}\right)^{\frac{-\rho}{\rho-1}} + 1}$$

Then the substitution effect is given by $x_1^s - x_1$ and the income effect is given by $x_1^{'} - x_1^s$.

3 Home Production

Bunter consumes two goods in quantities x_1 and x_2. Good 2 is a composite consumption good, and has price 1 per unit. Consumption of x_2 units of good 2 requires tx_2 units of time, so that t is the time cost per unit of good x_2. For example, time must be spent preparing food in order to consume it. Think of x_1 as *leisure* time that is not spent working and not spent fulflling the time-cost of consuming x_2. Bunter has a total of time T available, and earns w per unit of time spent working. All of time T is consumed either in x_1, working, or fulfilling the time cost of consuming x_2. Bunter's utility function is $u(x_1, x_2) = x_1 x_2^3$.

1. Solve Bunter's utility maximization problem to find his demand functions for goods 1 and 2.

   The problem of Bunter is the following:

   $$\underset{x_1, x_2, d, h}{\text{Max}} \left\{ x_1 x_2^3 \right\}$$

   $$\text{s.t. } x_2 \leq wh, tx_2 = d, d + h + x_1 = T$$

   where h is the time spent working and d is the time spent preparing food.

   The constraint $x_2 \leq wh$ says that the total amount spent on good 2 cannot exceed income, and income is obtained by working.

   The constraint $tx_2 = d$ says that the time spent cooking depends on the quantity of good 2 Bunter intends to consume.

   The last constraint says that the total amout of time spent cooking, loafing and working must be equal to the total time Bunter has.

   The problem can be simplified in order to have only two choice variables and only one constraint. In fact from the first constraint we get $h = \frac{x_2}{w}$ (to maximize utility Bunter spends all his income) and from the second constraint we get $d = tx_2$. Plugging in the third constraint we obtain

   $$\underset{x_1, x_2}{\text{Max}} \left\{ x_1 x_2^3 \right\}$$

   $$\text{s.t. } tx_2 + \frac{x_2}{w} + x_1 \leq T$$

and finally

$$\underset{x_1,x_2}{\text{Max}}\left\{x_1 x_2^3\right\}$$

$$\text{s.t. } \frac{1+tw}{w}x_2 + x_1 \leq T$$

Since we can eliminate all constraints, we must set up the Lagrangean:

$$L = x_1 x_2^3 + \lambda\left(T - \frac{1+tw}{w}x_2 - x_1\right)$$

Taking derivatives to get first order conditions

$$\frac{\partial L}{\partial x_1} = x_2^3 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 3x_1 x_2^2 - \lambda\frac{1+tw}{w} = 0$$

$$\frac{\partial L}{\partial \lambda} = T - \frac{1+tw}{w}x_2 - x_1 = 0$$

From the first 2 equations we get that $x_2 = 3\frac{w}{1+tw}x_1$.

Substituting into the third equation:

$$T = \frac{1+tw}{w}3\frac{w}{1+tw}x_1 + x_1 = 4x_1$$

$$\Rightarrow x_1 = \frac{T}{4}, x_2 = \frac{3}{4}\frac{w}{1+tw}T$$

2. Suppose t decreases. For example, new technologies maydecrease food preparation time. What is the effect on the consumptionof good 2? What is the effect on the amount of time this person spends working? Be precise.

   Clearly reducing t, x_2 increases. Working time is calculated via $h = \frac{x_2}{w} = \frac{3}{4}\frac{T}{1+tw}$. Increasing t reduces the time spent working (it's in the denominator).

3. Suppose the wage w increases. What is the effect onthe amount of time this person spends working, and on the amount ofleisure time, x_1 she consumes?

   From the expression for h we observe that an increase in the wage reduces time spent working.

   A change in wage has no effect on leisure time, which is a simple fraction of the total time available. The levels of t and w only affect the share of time allocated between working and preparing food.